Pensieve header: Testing the overhand equation.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\2012-04"];
<< betaCalculus.m
```

```
Clear[ℏ];
$PerturbativeDegree = 4;
βSimplify[expr_] := Replace[
   Series[Normal[expr], {ℏ, 0, $PerturbativeDegree}],
   sd_SeriesData :→ MapAt[Expand, sd, 3]
  ];
βCollect[B[ω_, μ_]] := B[
   βSimplify[ω],
   βSimplify[μ]
  ];
```

```
{V1, {sol}} = Get["VToDegree4-120420.m"];
```

```
V1
```

$$\begin{pmatrix} 1 - \frac{1}{16}(c_1 c_2)\hbar^2 + \left(\frac{c_1^3 c_2}{1728} + \frac{13 c_1^2 c_2^2}{4608} + \frac{c_1 c_2^3}{1728}\right)\hbar^4 + O[\hbar]^5 & \\ t[1] & -\frac{c_2 \hbar}{8} + \left(-\frac{1}{48} c_1 c_2 - \frac{c_2^2}{16}\right)\hbar^2 + \left(-\frac{7 c_1^2 c_2}{1728} - \frac{5}{384} c_1 \right. \\ t[2] & \frac{1}{2} + \left(\frac{c_1}{8} + \frac{c_2}{6}\right)\hbar + \left(\frac{c_1^2}{48} + \frac{c_1 c_2}{16} + \frac{c_2^2}{24}\right)\hbar^2 + \left(-\frac{37 c_1^2 c_2}{5760} + \frac{11}{19} \right. \end{pmatrix}$$

```
Φ1 = Φ[V1]
```

$$\begin{pmatrix} 1 & h[1] & \\ t[1] & \left(\frac{5 c_1 c_2 c_3}{1728} + \frac{23 c_2^2 c_3}{3456} - \frac{1}{576} c_2 c_3^2\right)\hbar^3 + O[\hbar]^5 & \frac{c_3 \hbar}{8} + \left(-\frac{67 c_1^2 c_3}{3456} - \frac{341 c}{8} \right. \\ t[2] & \frac{c_3 \hbar}{24} + \left(-\frac{311 c_1^2 c_3}{17280} - \frac{29}{864} c_1 c_2 c_3 - \frac{113 c_2^2 c_3}{17280} - \frac{23 c_1 c_3^2}{1152} - \frac{23 c_2 c_3^2}{2880} - \frac{11 c_3^3}{2880}\right)\hbar^3 + O[\hbar]^5 & \\ t[3] & \frac{c_2 \hbar}{24} + \left(-\frac{67 c_1^2 c_2}{5760} - \frac{1}{64} c_1 c_2^2 - \frac{7 c_2^3}{5760} + \frac{17 c_1 c_2 c_3}{2880} + \frac{19 c_2^2 c_3}{1440} + \frac{59 c_2 c_3^2}{5760}\right)\hbar^3 + O[\hbar]^5 & -\frac{c_1 \hbar}{8} + \left(\frac{c_1^3}{192} + \frac{5}{288} c_1^2 \right. \end{pmatrix}$$

The Overhand Equation:

```
Φ1 // dΔ[1, 0, 1] // dS[2] // dS[3] // dm[0, 3, 0] // dm[1, 2, 1]
```

$$\begin{pmatrix} 1 + \frac{1}{12} c_0 c_1 \hbar^2 + \left(-\frac{13 c_0^3 c_1}{2880} + \frac{29 c_0^2 c_1^2}{2880} + \frac{139 c_0 c_1^3}{17280}\right)\hbar^4 + O[\hbar]^5 & h[0] \\ t[0] & \left(-\frac{5 c_0 c_1^2}{1728} - \frac{17 c_1^3}{3456}\right)\hbar^3 + O[\hbar]^5 \\ t[1] & -\frac{c_1 \hbar}{12} + \left(\frac{4}{135} c_0^2 c_1 + \frac{547 c_0 c_1^2}{17280} + \frac{113 c_1^3}{17280}\right)\hbar^3 + O[\hbar]^5 \end{pmatrix}$$

```
{
 V0 = βCollect[
    B[ω[ℏ c1, ℏ c2], α[ℏ c1, ℏ c2] t[1] h[1] +
      β[ℏ c1, ℏ c2] t[1] h[2] + γ[ℏ c1, ℏ c2] t[2] h[1] + δ[ℏ c1, ℏ c2] t[2] h[2]]
   ] /. {
    (ϵ : (α | β | γ | δ | ω | κ))[____] :> ϵ0, (ϵ : (α | β | γ | δ | ω | κ))^(k___)[____] :> ϵFromDigits[{k}]
   },
 C0 = βCollect[B[κ[ℏ c1], 0]] /. {
    (ϵ : (α | β | γ | δ | ω | κ))[____] :> ϵ0, (ϵ : (α | β | γ | δ | ω | κ))^(k___)[____] :> ϵFromDigits[{k}]
   }
}
```

$$\left\{\begin{array}{l} \omega_0 + (c_2\,\omega_1 + c_1\,\omega_{10})\,\hbar + \left(\frac{1}{2}\,c_2^2\,\omega_2 + c_1\,c_2\,\omega_{11} + \frac{1}{2}\,c_1^2\,\omega_{20}\right)\hbar^2 + \left(\frac{1}{6}\,c_2^3\,\omega_3 + \frac{1}{2}\,c_1\,c_2^2\,\omega_{12} + \frac{1}{2}\,c_1^2\,c_2\,\omega_{21} + \frac{1}{6}\,c_1^3\,\omega_{30}\right. \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t[1] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t[2] \end{array}\right.$$

```
({V2, C2} = {V0, C0} /. sol) // ColumnForm
```

$$\left(\begin{array}{l} 1 + \frac{1}{16}\,c_1\,c_2\,(-1 + 16\,\delta_{10})\,\hbar^2 + \left(\frac{c_1^3\,c_2\,(1+48\,\beta_1+1152\,\beta_1^2-1152\,\beta_1\,\delta_{10}-576\,\delta_{10}^2+576\,\delta_{21})}{1728} + \frac{c_1\,c_2^3\,(1+48\,\beta_1+1152\,\beta_1^2-1152\,\beta_1\,\delta_{10}-576\,\delta}{1728}\right. \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t[1] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t[2] \end{array}\right.$$

$$\left(\begin{array}{l} 1 + c_1\,\kappa_1\,\hbar + \frac{1}{32}\,c_1^2\,(-1 + 16\,\delta_{10} + 16\,\kappa_1^2)\,\hbar^2 + \frac{1}{96}\,c_1^3\,\kappa_1\,(-3 + 48\,\delta_{10} + 16\,\kappa_1^2)\,\hbar^3 + \frac{c_1^4\,(35+384\,\beta_1+9216\,\beta_1^2-864\,\delta_{10}-921}{} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t[1] \end{array}\right.$$

```
Φ2 = Φ[V2]
```

$$\left(\begin{array}{l} 1 \\ t[1] \\ t[2] \quad \left(\frac{c_3}{24} + c_3\,\beta_1 - c_3\,\delta_{10}\right)\hbar + \left(-\frac{311\,c_1^2\,c_3}{17\,280} - \frac{29}{864}\,c_1\,c_2\,c_3 - \frac{113\,c_2^2\,c_3}{17\,280} - \frac{23\,c_1\,c_3^2}{1152} - \frac{23\,c_2\,c_3^2}{2880} - \frac{11\,c_3^3}{2880} - \frac{17}{36}\,c_1^2\,c_3\,\beta_1 - \frac{65}{72}\right. \\ t[3] \end{array}\right.$$

```
eqn = ((Φ2 // dΔ[1, 0, 1] // dS[2] // dS[3] // dm[0, 3, 0] // dm[1, 2, 1]) == B[1, 0])
```

$$1 + \left(\frac{c_0 c_1}{12} + c_0 c_1 \beta_1 - c_0 c_1 \delta_{10}\right) \hbar^2 + \left(-\frac{13 c_0^3 c_1}{2880} + \frac{29 c_0^2 c_1^2}{2880} + \frac{139 c_0 c_1^3}{17280} - \frac{1}{6} c_0^3 c_1 \beta_1 + \frac{1}{12} c_0^2 c_1^2 \beta_1 + \frac{1}{36} c_0 c_1^3 \beta_1 - c_0^3 c_1 \beta_1^2 - \frac{1}{3} c_0 c_1^3 \beta_1^2 - \frac{1}{2} c_0^2 c_1^2 \gamma_{12} - \frac{1}{2} c_0 c_1^3 \gamma_{12} + \frac{1}{6} c_0^3 c_1 \gamma_{30} + \frac{1}{2} c_0^2 c_1^2 \gamma_{30} + \frac{1}{2} c_0 c_1^3 \gamma_{30} + \frac{1}{24} c_0^3 c_1 \delta_{10} + \frac{1}{24} c_0^2 c_1^2 \delta_{10} + \frac{1}{12} c_0 c_1^3 \delta_{10} + c_0^3 c_1 \beta_1 \delta_{10} + c_0^2 c_1^2 \beta_1 \delta_{10} + \frac{4}{3} c_0 c_1^3 \beta_1 \delta_{10} - \frac{1}{3} c_0 c_1^3 \delta_{10}^2 + \frac{1}{2} c_0^2 c_1^2 \delta_{21} + \frac{1}{3} c_0 c_1^3 \delta_{21} - \frac{1}{6} c_0^3 c_1 \delta_{30} - \frac{1}{2} c_0^2 c_1^2 \delta_{30} - \frac{1}{3} c_0 c_1^3 \delta_{30}\right) \hbar^4 + O[\hbar]^5 == 1 \;\&\&$$

$$\hbar^3 \left(-\frac{5 c_0 c_1^2}{1728} - \frac{17 c_1^3}{3456} - \frac{1}{18} c_0 c_1^2 \beta_1 - \frac{1}{9} c_1^3 \beta_1 - \frac{1}{3} c_0 c_1^2 \beta_1^2 - \frac{2}{3} c_1^3 \beta_1^2 + \frac{1}{12} c_0 c_1^2 \delta_{10} + \frac{1}{8} c_1^3 \delta_{10} + \frac{4}{3} c_0 c_1^2 \beta_1 \delta_{10} + \frac{5}{3} c_1^3 \beta_1 \delta_{10} - \frac{1}{3} c_0 c_1^2 \delta_{10}^2 - \frac{2}{3} c_1^3 \delta_{10}^2 + \frac{1}{3} c_0 c_1^2 \delta_{21} + \frac{1}{6} c_1^3 \delta_{21} - \frac{1}{3} c_0 c_1^2 \delta_{30} - \frac{1}{6} c_1^3 \delta_{30}\right) == 0 \;\&\&$$

$$\hbar \left(-\frac{c_1}{12} - c_1 \beta_1 + c_1 \delta_{10}\right) + \hbar^3 \left(\frac{4}{135} c_0^2 c_1 + \frac{547 c_0 c_1^2}{17280} + \frac{113 c_1^3}{17280} + \frac{5}{9} c_0^2 c_1 \beta_1 + \frac{11}{18} c_0 c_1^2 \beta_1 + \frac{2}{9} c_1^3 \beta_1 + \frac{7}{3} c_0^2 c_1 \beta_1^2 + \frac{8}{3} c_0 c_1^2 \beta_1^2 + \frac{4}{3} c_1^3 \beta_1^2 - c_0^2 c_1 \gamma_{12} - c_0 c_1^2 \gamma_{12} + \frac{1}{2} c_0^2 c_1 \gamma_{30} + \frac{1}{2} c_0 c_1^2 \gamma_{30} - \frac{1}{6} c_1^3 \gamma_{30} + \frac{1}{24} c_0^2 c_1 \delta_{10} - \frac{1}{12} c_1^3 \delta_{10} - \frac{1}{3} c_0^2 c_1 \beta_1 \delta_{10} - \frac{2}{3} c_0 c_1^2 \beta_1 \delta_{10} - \frac{4}{3} c_1^3 \beta_1 \delta_{10} - \frac{2}{3} c_0^2 c_1 \delta_{10}^2 - \frac{1}{3} c_0 c_1^2 \delta_{10}^2 + \frac{1}{3} c_1^3 \delta_{10}^2 + \frac{2}{3} c_0^2 c_1 \delta_{21} + \frac{5}{6} c_0 c_1^2 \delta_{21} + \frac{1}{6} c_1^3 \delta_{21} - \frac{1}{6} c_0^2 c_1 \delta_{30} - \frac{1}{3} c_0 c_1^2 \delta_{30}\right) == 0 \;\&\&$$

$$\hbar \left(-\frac{c_1}{12} - c_1 \beta_1 + c_1 \delta_{10}\right) + \hbar^3 \left(\frac{4}{135} c_0^2 c_1 + \frac{547 c_0 c_1^2}{17280} + \frac{113 c_1^3}{17280} + \frac{5}{9} c_0^2 c_1 \beta_1 + \frac{11}{18} c_0 c_1^2 \beta_1 + \frac{2}{9} c_1^3 \beta_1 + \frac{7}{3} c_0^2 c_1 \beta_1^2 + \frac{8}{3} c_0 c_1^2 \beta_1^2 + \frac{4}{3} c_1^3 \beta_1^2 - c_0^2 c_1 \gamma_{12} - c_0 c_1^2 \gamma_{12} + \frac{1}{2} c_0^2 c_1 \gamma_{30} + \frac{1}{2} c_0 c_1^2 \gamma_{30} - \frac{1}{6} c_1^3 \gamma_{30} + \frac{1}{24} c_0^2 c_1 \delta_{10} - \frac{1}{12} c_1^3 \delta_{10} - \frac{1}{3} c_0^2 c_1 \beta_1 \delta_{10} - \frac{2}{3} c_0 c_1^2 \beta_1 \delta_{10} - \frac{4}{3} c_1^3 \beta_1 \delta_{10} - \frac{2}{3} c_0^2 c_1 \delta_{10}^2 - \frac{1}{3} c_0 c_1^2 \delta_{10}^2 + \frac{1}{3} c_1^3 \delta_{10}^2 + \frac{2}{3} c_0^2 c_1 \delta_{21} + \frac{5}{6} c_0 c_1^2 \delta_{21} + \frac{1}{6} c_1^3 \delta_{21} - \frac{1}{6} c_0^2 c_1 \delta_{30} - \frac{1}{3} c_0 c_1^2 \delta_{30}\right) == 0 \;\&\&$$

$$\hbar \left(\frac{c_0}{6} + 2 c_0 \beta_1 - 2 c_0 \delta_{10}\right) + \hbar^3 \left(-\frac{13 c_0^3}{1440} + \frac{139 c_0^2 c_1}{8640} + \frac{9}{640} c_0 c_1^2 - \frac{1}{3} c_0^3 \beta_1 + \frac{1}{18} c_0^2 c_1 \beta_1 - 2 c_0^3 \beta_1^2 - \frac{2}{3} c_0^2 c_1 \beta_1^2 - c_0 c_1^2 \beta_1^2 - c_0^2 c_1 \gamma_{12} - c_0 c_1^2 \gamma_{12} + \frac{1}{3} c_0^3 \gamma_{30} + c_0^2 c_1 \gamma_{30} + c_0 c_1^2 \gamma_{30} + \frac{1}{12} c_0^3 \delta_{10} + \frac{1}{6} c_0^2 c_1 \delta_{10} + \frac{5}{24} c_0 c_1^2 \delta_{10} + 2 c_0^3 \beta_1 \delta_{10} + \frac{8}{3} c_0^2 c_1 \beta_1 \delta_{10} + 3 c_0 c_1^2 \beta_1 \delta_{10} - \frac{2}{3} c_0^2 c_1 \delta_{10}^2 - c_0 c_1^2 \delta_{10}^2 + \frac{2}{3} c_0^2 c_1 \delta_{21} + \frac{1}{2} c_0 c_1^2 \delta_{21} - \frac{1}{3} c_0^3 \delta_{30} - \frac{2}{3} c_0^2 c_1 \delta_{30} - \frac{1}{2} c_0 c_1^2 \delta_{30}\right) == 0$$

```
sol1 = SolveAlways[eqn, {ℏ, c1, c2}];
Length /@ {sol1, sol1 = Union[Union /@ sol1]}
```

{27, 4}

```
sol1
```

$$\left\{\left\{c_0 \to 0,\ \beta_1 \to \frac{1}{12}(-1+12\delta_{10}),\ \gamma_{30} \to \frac{-47+480\delta_{10}+5760\delta_{10}^2+2880\delta_{21}}{2880},\ \delta_{30} \to \frac{1}{576}\left(-1-48\delta_{10}+1152\delta_{10}^2+576\delta_{21}\right)\right\},\right.$$
$$\left\{c_0 \to 0,\ \gamma_{30} \to \frac{1}{160}(1+40\beta_1+160\delta_{30}),\ \delta_{10} \to \frac{1}{12}(1+12\beta_1),\ \delta_{21} \to \frac{1}{192}\left(-1-48\beta_1-384\beta_1^2+192\delta_{30}\right)\right\},$$
$$\left\{\gamma_{12} \to \frac{1}{960}(27+400\beta_1+960\delta_{30}),\ \gamma_{30} \to \frac{1}{160}(1+40\beta_1+160\delta_{30}),\ \delta_{10} \to \frac{1}{12}(1+12\beta_1),\ \delta_{21} \to \frac{1}{192}\left(-1-48\beta_1-384\beta_1^2+192\delta_{30}\right)\right\},$$
$$\left.\left\{c_0 \to 0,\ \gamma_{12} \to \frac{1}{30}\left(1+20\beta_1+60\beta_1^2+30\delta_{21}\right),\ \gamma_{30} \to \frac{1}{960}\left(11+480\beta_1+1920\beta_1^2+960\delta_{21}\right),\ \delta_{10} \to \frac{1}{12}(1+12\beta_1),\ \delta_{30} \to \frac{1}{192}\left(1+48\beta_1+384\beta_1^2+192\delta_{21}\right)\right\}\right\}$$

```
indvars = Union[Cases[Last /@ #, ϵ_k_ :> ϵk, Infinity]] & /@ sol1
```

$\{\{\delta_{10}, \delta_{21}\}, \{\beta_1, \delta_{30}\}, \{\beta_1, \delta_{30}\}, \{\beta_1, \delta_{21}\}\}$

```
({V3, C3} = {V2, C2} /. sol1[[1]] /. Thread[{δ10, δ21} → 0]) // ColumnForm
```

$$\begin{pmatrix} 1-\frac{1}{16}(c_1 c_2)\hbar^2+\left(\frac{5c_1^3c_2}{1728}+\frac{29c_1^2c_2^2}{4608}+\frac{5c_1c_2^3}{1728}\right)\hbar^4+O[\hbar]^5 & -\frac{c_2\hbar}{8}+\left(\frac{c_1c_2}{48}-\frac{c_2^2}{48}\right)\hbar^2+ \\ t[1] & \\ t[2] & \frac{1}{2}+\left(\frac{c_1}{24}+\frac{c_2}{12}\right)\hbar+\left(-\frac{c_1^2}{48}-\frac{c_1c_2}{48}\right)\hbar^2+\left(-\frac{47c_1^3}{17280}-\frac{17c}{57}\right. \end{pmatrix}$$

$$\begin{pmatrix} 1+c_1\kappa_1\hbar+\frac{1}{32}c_1^2(-1+16\kappa_1^2)\hbar^2+\frac{1}{96}c_1^3\kappa_1(-3+16\kappa_1^2)\hbar^3+\frac{c_1^4(67-864\kappa_1^2+2304\kappa_1^4)\hbar^4}{55296}+O[\hbar]^5 \\ t[1] \end{pmatrix}$$

The third non-degeneracy equation for V:

```
V2 // dS[2] // dm[1, 2, 1]
```

$$\begin{pmatrix} 1+\frac{1}{48}c_1^2\hbar^2+\frac{c_1^4\hbar^4}{23040}+O[\hbar]^5 & h[1] \\ t[1] & -\frac{1}{2}+\frac{c_1\hbar}{8}-\frac{1}{48}c_1^2\hbar^2+\frac{1}{384}c_1^3\hbar^3-\frac{c_1^4\hbar^4}{3840}+O[\hbar]^5 \end{pmatrix}$$

```
(V2 /. sol1[[1]]) // dS[2] // dm[1, 2, 1]
```

$$\begin{pmatrix} 1+\frac{1}{48}c_1^2\hbar^2+\frac{c_1^4\hbar^4}{23040}+O[\hbar]^5 & h[1] \\ t[1] & -\frac{1}{2}+\frac{c_1\hbar}{8}-\frac{1}{48}c_1^2\hbar^2+\frac{1}{384}c_1^3\hbar^3-\frac{c_1^4\hbar^4}{3840}+O[\hbar]^5 \end{pmatrix}$$

```
R[1, 1, -1 / 2]
```

$$\begin{pmatrix} 1 & h[1] \\ t[1] & -\frac{1}{2}+\frac{c_1\hbar}{8}-\frac{1}{48}c_1^2\hbar^2+\frac{1}{384}c_1^3\hbar^3-\frac{c_1^4\hbar^4}{3840}+O[\hbar]^5 \end{pmatrix}$$

```
C3 /. {κ1 → 0}
```

$$\begin{pmatrix} 1 - \frac{1}{32} c_1^2 \hbar^2 + \frac{67 c_1^4 \hbar^4}{55296} + O[\hbar]^5 \\ t[1] \end{pmatrix}$$

```
((V2 /. sol1[[1]]) // dS[2] // dm[1, 2, 1]) ** R[1, 1, 1 / 2]
FactorInteger[23040]
```

{{2, 9}, {3, 2}, {5, 1}}

```
23040 / 5! / 2^5
```

6

```
R[1, 1, -1 / 2] // dS[1]
```

$$\begin{pmatrix} 1 - \frac{c_1 \hbar}{2} + \frac{1}{8} c_1^2 \hbar^2 - \frac{1}{48} c_1^3 \hbar^3 + \frac{1}{384} c_1^4 \hbar^4 + O[\hbar]^5 & h[1] \\ t[1] & -\frac{1}{2} + \frac{c_1 \hbar}{8} - \frac{1}{48} c_1^2 \hbar^2 + \frac{1}{384} c_1^3 \hbar^3 - \frac{c_1^4 \hbar^4}{3840} + O[\hbar]^5 \end{pmatrix}$$

```
R[1, 2] // dS[2] // dm[1, 2, 1]
```

$$\begin{pmatrix} 1 - c_1 \hbar + \frac{1}{2} c_1^2 \hbar^2 - \frac{1}{6} c_1^3 \hbar^3 + \frac{1}{24} c_1^4 \hbar^4 + O[\hbar]^5 & h[1] \\ t[1] & -1 + \frac{c_1 \hbar}{2} - \frac{1}{6} c_1^2 \hbar^2 + \frac{1}{24} c_1^3 \hbar^3 - \frac{1}{120} c_1^4 \hbar^4 + O[\hbar]^5 \end{pmatrix}$$

```
R[2, 1] // dS[2] // dm[1, 2, 1]
```

$$\begin{pmatrix} 1 & h[1] \\ t[1] & -1 + \frac{c_1 \hbar}{2} - \frac{1}{6} c_1^2 \hbar^2 + \frac{1}{24} c_1^3 \hbar^3 - \frac{1}{120} c_1^4 \hbar^4 + O[\hbar]^5 \end{pmatrix}$$

```
{R[1, 1], R[1, 1] // dS[1]} // ColumnForm
```

$$\begin{pmatrix} 1 & h[1] \\ t[1] & 1 + \frac{c_1 \hbar}{2} + \frac{1}{6} c_1^2 \hbar^2 + \frac{1}{24} c_1^3 \hbar^3 + \frac{1}{120} c_1^4 \hbar^4 + O[\hbar]^5 \end{pmatrix}$$

$$\begin{pmatrix} 1 + c_1 \hbar + \frac{1}{2} c_1^2 \hbar^2 + \frac{1}{6} c_1^3 \hbar^3 + \frac{1}{24} c_1^4 \hbar^4 + O[\hbar]^5 & h[1] \\ t[1] & 1 + \frac{c_1 \hbar}{2} + \frac{1}{6} c_1^2 \hbar^2 + \frac{1}{24} c_1^3 \hbar^3 + \frac{1}{120} c_1^4 \hbar^4 + O[\hbar]^5 \end{pmatrix}$$

```
R[1, 1] == (R[1, 2] // dm[2, 1, 1])
```

True

```
Er = (V2 /. sol1[[1]]) // dS[2] // dm[1, 2, 1]
```

$$\begin{pmatrix} 1 + \frac{1}{48} c_1^2 \hbar^2 + \frac{c_1^4 \hbar^4}{23040} + O[\hbar]^5 & h[1] \\ t[1] & -\frac{1}{2} + \frac{c_1 \hbar}{8} - \frac{1}{48} c_1^2 \hbar^2 + \frac{1}{384} c_1^3 \hbar^3 - \frac{c_1^4 \hbar^4}{3840} + O[\hbar]^5 \end{pmatrix}$$

```
23040 / 1920
```

12

$$1 \Big/ \left(1 + \frac{1}{48} c_1^2 \hbar^2 + \frac{c_1^4 \hbar^4}{23040} + O[\hbar]^5\right)$$

$$1 - \frac{1}{48} c_1^2 \hbar^2 + \frac{c_1^4 \hbar^4}{2560} + O[\hbar]^5$$

```
1920 / 2560
```

$$\frac{3}{4}$$

```
Er ** (Er // dS[1])
```

$$\begin{pmatrix} 1 - \frac{c_1 \hbar}{2} + \frac{1}{6} c_1^2 \hbar^2 - \frac{1}{24} c_1^3 \hbar^3 + \frac{1}{120} c_1^4 \hbar^4 + O[\hbar]^5 & h[1] \\ t[1] & -1 + \frac{c_1 \hbar}{2} - \frac{1}{6} c_1^2 \hbar^2 + \frac{1}{24} c_1^3 \hbar^3 - \frac{1}{120} c_1^4 \hbar^4 + O[\hbar]^5 \end{pmatrix}$$

```
R[1, 1, -1] // dS[1]
```

$$\begin{pmatrix} 1 - c_1 \hbar + \frac{1}{2} c_1^2 \hbar^2 - \frac{1}{6} c_1^3 \hbar^3 + \frac{1}{24} c_1^4 \hbar^4 + O[\hbar]^5 & h[1] \\ t[1] & -1 + \frac{c_1 \hbar}{2} - \frac{1}{6} c_1^2 \hbar^2 + \frac{1}{24} c_1^3 \hbar^3 - \frac{1}{120} c_1^4 \hbar^4 + O[\hbar]^5 \end{pmatrix}$$

```
Θ[1, 2] == (Θ[1, 2] // dS[1] // dS[2])
```

True

```
(Er // dS[1]) ** Er
```

$$\begin{pmatrix} 1 - \frac{c_1 \hbar}{2} + \frac{1}{6} c_1^2 \hbar^2 - \frac{1}{24} c_1^3 \hbar^3 + \frac{1}{120} c_1^4 \hbar^4 + O[\hbar]^5 & h[1] \\ t[1] & -1 + \frac{c_1 \hbar}{2} - \frac{1}{6} c_1^2 \hbar^2 + \frac{1}{24} c_1^3 \hbar^3 - \frac{1}{120} c_1^4 \hbar^4 + O[\hbar]^5 \end{pmatrix}$$

```
((Er // dS[1]) ** Er ) // dcap[1]
```

$$\begin{pmatrix} 1 + \frac{c_1 \hbar}{2} + \frac{1}{6} c_1^2 \hbar^2 + \frac{1}{24} c_1^3 \hbar^3 + \frac{1}{120} c_1^4 \hbar^4 + O[\hbar]^5 \\ t[1] \end{pmatrix}$$

```
{ser = Series[(E^x - 1) / x, {x, 0, 8}], Exp[ser], Log[ser]} // ColumnForm
```

$$1 + \frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \frac{x^4}{120} + \frac{x^5}{720} + \frac{x^6}{5040} + \frac{x^7}{40\,320} + \frac{x^8}{362\,880} + O[x]^9$$

$$e + \frac{e\,x}{2} + \frac{7\,e\,x^2}{24} + \frac{7\,e\,x^3}{48} + \frac{383\,e\,x^4}{5760} + \frac{109\,e\,x^5}{3840} + \frac{33\,371\,e\,x^6}{2\,903\,040} + \frac{25\,859\,e\,x^7}{5\,806\,080} + \frac{2\,315\,239\,e\,x^8}{1\,393\,459\,200} + O[x]^9$$

$$\frac{x}{2} + \frac{x^2}{24} - \frac{x^4}{2880} + \frac{x^6}{181\,440} - \frac{x^8}{9\,676\,800} + O[x]^9$$

```
(Inverse[V2] (V2 // dP[3, 4])) // dm[2, 3, 2] // dS[4] // dΔ[4, 4, 5] // dm[4, 1, 1] //
 dm[5, 2, 2]
```

$$\left(1 - \frac{c_2 \hbar}{2} + \left(\frac{c_1 c_2}{24} + \frac{5 c_2^2}{48} + c_1 c_2 \beta_1 - c_1 c_2 \delta_{10} \right) \hbar^2 + \left(- \frac{1}{48} c_1 c_2^2 - \frac{c_2^3}{96} - \frac{1}{2} c_1 c_2^2 \beta_1 + \frac{1}{2} c_1 c_2^2 \delta_{10} \right) \hbar^3 + \left(- \frac{11 c_1^3 c_2}{2880} + \right.$$

```
((Inverse[V3] (V3 // dP[3, 4])) // dm[2, 3, 2] // dS[4] // dΔ[4, 4, 5] // dm[4, 1, 1] //
   dm[5, 2, 2]) /. γ12 → -1 / 1920
```

$$\left(1 - \frac{c_2 \hbar}{2} + \left(- \frac{1}{24} c_1 c_2 + \frac{5 c_2^2}{48} \right) \hbar^2 + \left(\frac{1}{48} c_1 c_2^2 - \frac{c_2^3}{96} \right) \hbar^3 + \left(\frac{c_1^3 c_2}{1440} - \frac{23 c_1^2 c_2^2}{11\,520} - \frac{29 c_1 c_2^3}{3840} + \frac{c_2^4}{2560} \right) \hbar^4 + O[\hbar]^5 \right.$$

$$t[1] \qquad - \frac{c_2}{}$$

$$t[2] \qquad - \frac{1}{2} + \left(\frac{c_1}{8} \right.$$

```
Φ1 // dcap[1] // dcap[2] // dcap[3]
```

(1)

```
Φ1 // Inverse // dcap[1] // dcap[2] // dcap[3]
```

(1)